Stochastic Processes and their Applications 24 (1987) 293-307
North-Holland

THE FULL-INFORMATION BEST CHOICE PROBLEM WITH A RANDOM NUMBER OF OBSERVATIONS

Zdzisław POROSIŃSKI

Institute of Mathematics, Technical University of Wrocław, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

Received 10 December 1985
Revised 16 January 1987

The full-information best choice problem with a random number of observations is considered. N i.i.d. random variables with a known continuous distribution are observed sequentially with the object of selecting the largest. Neither recall nor uncertainty of selection is allowed and one choice must be made. In this paper the number N of observations is random with a known distribution. The structure of the stopping set is investigated. A class of distributions of N (which contains in particular the uniform, negative-binomial and Poisson distributions) is determined, for which the so-called "monotone case" occurs. The theoretical solution for the monotone case is considered. In the case where N is geometric the optimal solution is presented and the probability of winning worked out. Finally, the case where N is uniform is examined. A simple asymptotically optimal stopping rule is found and the asymptotic probability of winning is obtained.

optimal stopping * best choice problem * secretary problem

1. Introduction

The following best choice problem was studied by Gilbert and Mosteller [6, Section 3]. A known number, N, of i.i.d. random variables from a known continuous distribution F are observed sequentially. The objective is to maximize the probability of selecting the largest. Neither recall of observations nor uncertainty of selection is allowed and one choice must be made.

This problem for a finite number of observations was solved by a heuristic argument by Gilbert and Mosteller in [6]. Bojdecki [1] has given the rigorous proof of this result.

In this paper a similar problem is considered in which the number N of observations is allowed to be a random variable with a known distribution. In this generalization the influence of various factors on the length of the observation is taken into account. Since N is unknown, the observer faces an additional risk. If he rejects any observation, he may then discover it was the last one, in which case he receives nothing at all.

The organization of the present paper is as follows. In Section 2 the precise formulation of the above problem is given. This problem can be lead to the classical

Research supported by Central Basic Research Program under contract CPBP 01.02./IV.2.

optimal stopping problem for some Markov chain (cf. Shiryaev [9]). In Section 2 this reduction is presented too. Section 3 contains preliminary definitions and the next modification which facilitates the examination of the stopping set Δ. Properties of Δ in the general case are investigated in Section 4. The structure of the stopping set is described in Theorem 1 via the number of changes of sign of certain sequences.

Theorem 1 is effectively used in Section 5 in which the so-called "monotone case" is considered. Several examples are given and Theorem 2 giving the solution in this case is proved. The optimal stopping rule can be expressed as follows: stop the observation at the moment n in which the first "leader" occurs such that it exceeds x_n, where the sequence of the optimal decision numbers $(x_n)_{n=1}^{\infty}$ is non-increasing.

In Section 6 special examples when the period of observation has the geometric or the uniform distribution are considered in detail. In the geometric case it is interesting and quite unexpected that the probability of winning (the correct stop) in all "natural" situations is constant, equal to e^{-1}. In the uniform case the optimal decision numbers for $n=1(1)60$ and the probability of winning for $n=1(1)10(5)20(10)60$ are given. A simple asymptotically optimal stopping rule is found and the asymptotic probability of winning (equal to 0.4352) is obtained.

2. The model and its reduction

Assume that

(1) $\xi_1, \xi_2, \xi_3, \ldots$ is a sequence of i.i.d. random variables with a continuous distribution function F, defined on the probability space $(\Omega, \mathcal{F}, P)$, and

(2) the number of observations N is a random variable independent of the sequence $(\xi_n)_{n=1}^{\infty}$ with a known distribution

$$P(N=n)=p_n, \quad n=0,1,2,\ldots, \qquad \sum_{n=0}^{\infty} p_n = 1.$$

Let $\mathcal{T}$ be the set of all Markov moments with respect to the family of σ-fields $(\mathcal{F}_n)_{n=1}^{\infty}$, where $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n, I_{\{0\}}(N), \ldots, I_{\{n-1\}}(N))$ and I_A denotes the indicator function of the event A.

Consider the following problem:

(P) Find a stopping time $\tau^* \in \mathcal{T}$ such that

$$P(\tau^* \leq N, \xi_{\tau^*} = \max(\xi_1, \ldots, \xi_N)) = \sup_{\tau \in \mathcal{T}} P(\tau \leq N, \xi_\tau = \max(\xi_1, \ldots, \xi_N)).$$

Since F is known and continuous then without loss of generality we may additionally assume that

(3) ξ_n has the uniform distribution on $[0,1]$, $n \in \mathbb{N} = \{1,2,\ldots\}$. Denote

$$\begin{aligned} Z_n &= P(N \geq n, \xi_n = \max(\xi_1, \ldots, \xi_N) \mid \mathcal{F}_n) \\ &= I_{\{\xi_n = \max(\xi_1,\ldots,\xi_n)\}} \sum_{m=n}^{\infty} P(N=m, \xi_n = \max(\xi_n, \ldots, \xi_m) \mid \mathcal{F}_n) \\ &= I_{\{\xi_n = \max(\xi_1,\ldots,\xi_n)\}} W_n \end{aligned}$$

where

$$W_n = \sum_{m=n}^{\infty} (p_m/\pi_n)\xi_n^{m-n}, \quad \pi_n = \sum_{i=n}^{\infty} p_i$$

and $Z_\infty = 0$. Hence $E(Z_\tau) = P(\tau \leq N, \xi_\tau = \max(\xi_1, \ldots, \xi_N))$.

It suffices to consider Markov moments belonging to the set of "leaders"

$$\mathcal{T}_0 = \{\tau \in \mathcal{T}\colon \tau = n \Rightarrow \xi_n = \max(\xi_1, \ldots, \xi_n),\ n \in \mathbb{N}\}$$

only. Now let

$$\tau_1 = \begin{cases} 1 & \text{if } N \geq 1, \\ +\infty & \text{if } N = 0; \end{cases}$$

$$\tau_{k+1} = \inf\{n\colon n > \tau_k,\ n \leq N,\ \xi_n = \max(\xi_1, \ldots, \xi_n)\}, \quad k \in \mathbb{N}.$$

Define, for $k \in \mathbb{N}$, $Y_k = (\tau_k, \xi_{\tau_k})$ if $\tau_k < +\infty$ and $Y_k = \delta$ if $\tau_k = +\infty$, where δ is a label for the final state. $Y = (Y_k)_{k=1}^{\infty}$ is a homogeneous Markov chain with respect to the σ-fields $(\mathcal{F}_{\tau_k})_{k=1}^{\infty}$. The state space of this chain is $\mathbb{E} = \mathbb{N} \times [0, 1] \cup \{\delta\}$. Since, for $k \in \mathbb{N}$,

$$P(Y_{k+1} \in \{m\} \times [0, y] \mid \mathcal{F}_{\tau_k}) = \sum_{n=1}^{m-1} I_{\{\tau_k = n\}} P(\tau_{k+1} = m,\ \xi_m \leq y \mid \mathcal{F}_n)$$

$$= \begin{cases} \sum_{n=1}^{m-1} I_{\{\tau_k = n\}} (\pi_m/\pi_n)\xi_n^{m-n-1}(y - \xi_n) & \text{if } y \geq \xi_n, \\ 0 & \text{if } y < \xi_n, \end{cases}$$

the transition function is

$$p(n, x; m, [0, y]) = P(\tau_{k+1} = m,\ \xi_m \leq y \mid \tau_k = n,\ \xi_n = x)$$

$$= \begin{cases} (\pi_m/\pi_n)x^{m-n-1}(y - x) & \text{if } n < m \text{ and } x \leq y, \\ 0 & \text{otherwise,} \end{cases}$$

$$p(n, x; \delta) = \sum_{m=n}^{\infty} (p_m/\pi_n)x^{m-n}, \qquad p(\delta, \delta) = 1. \tag{4}$$

For any $\tau \in \mathcal{T}_0$ we define a Markov moment σ with respect to $(\mathcal{F}_{\tau_k})_{k=1}^{\infty}$ as follows: $\sigma = k$ on the set $\{\tau = \tau_k < +\infty\}$, $k \in \mathbb{N}$, and $\sigma = +\infty$ on $\{\tau = +\infty\}$. Thus we have

$$Z_\tau = \left.\begin{cases} W_{\tau_\sigma} & \text{if } \tau < +\infty \\ 0 & \text{if } \tau = +\infty \end{cases}\right\} = f_0(Y_\sigma)$$

where $f_0(\delta) = 0$ ($Y_\infty = \delta$ by definition) and

$$f_0(n, x) = \sum_{m=n}^{\infty} (p_m/\pi_n)x^{m-n} \quad \text{for } n \in \mathbb{N},\ x \in [0, 1].$$

Thus we reduce the initial Problem (P) to the problem of optimal stopping of the Markov chain Y with the reward function f_0.

3. Preliminaries

Now one should calculate

$$s_0(n, x) = \sup_{\tau \in \mathcal{T}, \tau \geq n} E_{(n,x)} f_0(\tau, \xi_\tau),$$

where $E_{(n,x)}$ denotes the expected value with respect to $P_{(n,x)}(\cdot) = p(n, x; \cdot)$, and display a certain optimal τ. From the general theory of optimal stopping (cf. e.g. Shiryaev [9]) we have that $s_0(n, x)$ satisfies

$$s_0(n, x) = \max\{f_0(n, x), \mathbb{P}_0 s_0(n, x)\} \tag{5}$$

where

$$\mathbb{P}_0 h(e) = \int_{\mathbb{E}} h(a) P_e(\mathrm{d}a) \tag{6}$$

for a bounded function $h : \mathbb{E} \to \mathbb{R} = (-\infty, +\infty)$. So $\mathbb{P}_0 h(\delta) = 0$ and, from (4),

$$\mathbb{P}_0 h(n, x) = \sum_{m=n+1}^{\infty} \int_x^1 h(m, y)(\pi_m / \pi_n) x^{m-n-1} \, \mathrm{d}y.$$

We do not consider the component $h(\delta)p(n, x; \delta)$ because $f_0(\delta) = 0$, $p(\delta, \delta) = 1$, $s_0(\delta) = 0$ and we additionally assume that $h(\delta) = 0$.

The set $\Delta = \{e \in \mathbb{E}: s_0(e) = f_0(e)\}$ we call the stopping set. It is well known that the Markov moment $\tau_0 = \inf\{n \in \mathbb{N}: Y_n \in \Delta\}$ is optimal only if $\tau_0 < \infty$ almost surely (a.s.).

Our chain attains the state δ a.s.; therefore τ_0 is optimal for Problem (P) and in order to calculate the optimal strategy it suffices to investigate the stopping set. For such a family of functions h (i.e. bounded and $h(\partial) = 0$), the equality (5) has exactly one solution which completely characterizes the set Δ. Denote

$$\begin{aligned} f(n, x) &= \pi_n f_0(n, x) = \sum_{m=n}^{\infty} p_m x^{m-n}, \\ s(n, x) &= \pi_n s_0(n, x). \end{aligned} \tag{7}$$

Now we transform (5) to

$$s(n, x) = \max\{f(n, x), \mathbb{P} s(n, x)\} \tag{8}$$

where

$$\mathbb{P} s(n, x) = \sum_{m=n+1}^{\infty} \int_x^1 s(m, y) x^{m-n-1} \, \mathrm{d}y. \tag{9}$$

It is well known that $s_0(n, x) = \lim_{k \to \infty} \mathbb{Q}_0^k f_0(n, x)$, where $\mathbb{Q}_0 f_0(n, x) = \max\{f_0(n, x), \mathbb{P}_0 f_0(n, x)\}$. This equality can be written as

$$s(n, x) = \lim_{k \to \infty} \mathbb{Q}^k f(n, x)$$

where

$$\mathbb{Q}f(n,x)=\max\{f(n,x),\mathbb{P}f(n,x)\}.$$

Moreover, it is easy to see that

$$\Delta=\{(n,x)\colon s(n,x)=f(n,x)\}\cup\{\delta\}. \tag{10}$$

From (8) and (10) we have

$$\begin{aligned}(n,x)\in\Delta &\Leftrightarrow s(n,x)=f(n,x)\geqslant\mathbb{P}s(n,x),\\ (n,x)\notin\Delta &\Leftrightarrow s(n,x)=\mathbb{P}s(n,x)>f(n,x).\end{aligned} \tag{11}$$

Note also that

$$\begin{aligned}f(n,x)&=p_n+xf(n+1,x),\\ \mathbb{P}s(n,x)&=\int_x^1 s(n+1,y)\,\mathrm{d}y+x\mathbb{P}s(n+1,x).\end{aligned} \tag{12}$$

4. General properties

In the general case it is very difficult to determine the stopping set. Now we investigate some properties of Δ.

First note that $\delta\in\Delta$ because $f(\delta)=0=s(\delta)$. Moreover, since for each fixed n the function $f(n,x)$ is non-decreasing, the function $\mathbb{P}s(n,x)$ is non-increasing and $f(n,1)=\pi_n$, $\mathbb{P}s(n,1)=0$, there exists $x_n\in[0,1]$ such that $\Delta\cap\{n\}\times[0,1]=\{n\}\times[x_n,1]$. Thus we can write

$$\Delta=\{\delta\}\cup\bigcup_{n=1}^{\infty}(\{n\}\times[x_n,1]). \tag{13}$$

Denote by

$$\begin{aligned}c(n,x)&=f(n,x)-\mathbb{P}f(n,x)\\ &=\sum_{m=n}^{\infty}p_m x^{m-n}-\sum_{m=n+1}^{\infty}x^{m-n-1}\int_x^1 f(m,y)\,\mathrm{d}y=\sum_{m=n}^{\infty}x^{m-n}\,\mathrm{d}(m,x)\end{aligned} \tag{14}$$

where

$$d(m,x)=p_m-\int_x^1 f(m+1,y)\,\mathrm{d}y \tag{15}$$

for $n=0,1,2,\ldots$ and $x\in[0,1]$.

The structure of Δ depends on values of $c(n,x)$ (it is obvious that $c(n,x)\geqslant 0$ for $(n,x)\in\Delta$), but the investigation of $c(n,x)$ is difficult because of a form of this function. Simpler conditions can be obtained investigating the function $d(n,x)$ given by (15). To give the structure of the stopping set it suffices to know the structure of sets $\Delta(x)=\{n\in\mathbb{N}: (n, x)\in\Delta\}$, $x\in[0,1]$. Following Presman and Sonin we give two definitions which will be used in Theorem 1 below. Let x be fixed. If k, $k+1,\ldots,m\in\Delta(x)$ and $k-1$, $m+1\notin\Delta(x)$ then the set $\{k,\ldots,m\}$ is called the stopping island (obviously $m\leq\infty$). We say that the sequence $(d(n,x))_{n=-1}^{\infty}$ changes sign at the point m if $d(m,x)\geq 0$ and $d(m-1,x)<0$ (for the convenience of the definition we adopt $d(-1,x)=-1$).

Theorem 1. *If the sequence $(d(n,x))_{n=-1}^{\infty}$ changes sign M times then $\Delta(x)$ has no more than M stopping islands.*

First we prove the following lemmas.

Lemma 1. *If $m\in\Delta(x)$ and $d(m-1,x)\geq 0$ then $m-1\in\Delta(x)$.*

Proof. Since $s(m,y)=f(m,y)$ for all $y\geq x$ and $d(m-1,x)\geq 0$,

$$d(m-1,x)+\int_x^1 f(m,y)\,dy\geq\int_x^1 s(m,y)\,\mathrm{d}y.$$

From (15) this inequality is equivalent to $p_{m-1}\geq\int_x^1 s(m,y)\,\mathrm{d}y$ and, from (11), $f(m,x)\geq\mathbb{P}s(m,x)$. Therefore

$$p_{m-1}+xf(m,x)\geq\int_x^1 s(m,y)\,\mathrm{d}y+x\mathbb{P}s(m,x),$$

which from (12) is equivalent to $f(m-1,x)\geq\mathbb{P}s(m-1,x)$. Again (11) implies $m-1\in\Delta(x)$.

Lemma 2. *If $m\in\Delta(x)$ and $d(m,x)\leq 0$ then $m+1\in\Delta(x)$.*

Proof. Suppose $m+1\notin\Delta(x)$. We must show $m\notin\Delta(x)$ or $d(m,x)>0$. Using (12), (15) and (11) we obtain

$$\begin{aligned} f(m,x)=p_m+xf(m+1,x)&=d(m,x)+\int_x^1 f(m+1,y)\,\mathrm{d}y+xf(m+1,x)\\ &<d(m,x)+\int_x^1 s(m+1,y)\,\mathrm{d}y+xf(m+1,x)\\ &=d(m,x)+\mathbb{P}s(m,x)+x[f(m+1,x)-\mathbb{P}s(m+1,x)]\\ &<d(m,x)+\mathbb{P}s(m,x). \end{aligned}\tag{16}$$

If $d(m,x)\leq 0$ then $f(m,x)-\mathbb{P}s(m,x)<d(m,x)\leq 0$ and (11) implies $m\notin\Delta(x)$. If $m\in\Delta(x)$ then, again from (11), $d(m,x)>f(m,x)-\mathbb{P}s(m,x)\geq 0$ and Lemma 2 is proved.

Proof of Theorem 1. Let x be fixed. Suppose $\{m, \ldots, n\}$ is a stopping island. From Lemma 2 we have $d(n, x) > 0$. Lemma 1 and the convention $d(-1, x) = -1$ imply $d(m-1, x) < 0$ and therefore the sequence $(d(k, x))_{k=m-1}^{n}$ changes sign at least once. If a stopping island has a form $\{m, m+1, \ldots\}$, then analogously to above, $d(m-1, x) < 0$ and $c(m, x) \geq 0$. But from (14), $c(m, x) \geq 0$ implies that $d(n, x) \geq 0$ for some $n \geq m$. Therefore the sequence $(d(n, x))_{n=m-1}^{\infty}$ also changes sign at least once in this case. This completes the proof.

5. The monotone case. Examples

Theorem 1 can be especially useful if $M = 1$ for each $x \in [0, 1]$. Note that then we obtain the Markov version of the "monotone case" (cf. [4]). Many frequently used probability distributions $(p_n)_{n=0}^{\infty}$ give $M = 1$. In the examples given below x is a fixed number in $[0, 1)$; notice that always $\mathbb{N} \times \{1\} = \Delta(1)$.

Example 1. *The one-point distribution.* If $P(N = n) = 1$ then $d(k, x) < 0$ for $k < n$, $d(n, x) > 0$ and $d(k, x) = 0$ for $k > n$. Thus $(d(k, x))_{k=-1}^{\infty}$ changes sign exactly once. This case is the well-known full-information best choice problem considered by Gilbert and Mosteller [6].

Example 2. *The uniform distribution on* $\{1, 2, \ldots, n\}$. For $1 \leq k \leq n$ we have from (15) and (7) that

$$d(k, x) = \frac{1}{n}\left(1 - \int_x^1 \sum_{i=k+1}^{n} y^{i-(k+1)}\, \mathrm{d}y\right) = \frac{1}{n}\left(1 - \int_x^1 \sum_{i=0}^{n-k-1} y^i\, \mathrm{d}y\right).$$

Since $d(0, x) < 0$, $d(n, x) = 1/n$, the sequence $(d(k, x))_{k=1}^{n}$ is increasing, $d(k, x) = 0$ for $k > n$, and then $M = 1$. This interesting example will be examined later.

Example 3. *The Poisson distribution with the parameter* λ. Here

$$d(k, x) = \frac{\lambda^k}{k!} \mathrm{e}^{-\lambda} - \int_x^1 \sum_{i=k+1}^{\infty} \frac{\lambda^i}{i!} \mathrm{e}^{-\lambda} y^{i-(k+1)}\, \mathrm{d}y = \frac{\lambda^k}{k!} \mathrm{e}^{-\lambda}(1 - a(k, x))$$

where

$$a(k, x) = \sum_{i=1}^{\infty} \frac{k!\,\lambda^i}{(i+k)!\,i}(1 - x^i).$$

Since $a(k, x)$ is decreasing (as is easy to show) we obtain that $d(k, x)$ changes sign once.

Example 4. *The negative-binomial distribution with parameters p, r.*

If $k<r$ then $d(k,x)<0$. Now let $k\geqslant r$. Then,

$$d(k,x)=\binom{k-1}{r-1}p^rq^{k-r}-\int_x^1\sum_{i=k+1}^{\infty}\binom{i-1}{r-1}p^rq^{i-r}y^{i-(k+1)}\,\mathrm{d}y$$

$$=\binom{k-1}{r-1}p^rq^{k-r}(1-b(k,x))$$

where

$$b(k,x)=\left[\binom{k-1}{r-1}\right]^{-1}\sum_{i=1}^{\infty}\binom{k+i-1}{k+i-r}\frac{q^i}{i}(1-x^i).$$

Since, for $r>1$, $b(k,x)$ decreases if k increases, then $d(k,x)$ changes sign once. When $r=1$, i.e. for the geometric distribution with parameter p, the sequence $(b(k,x))_{k=0}^{\infty}$ is constant and the above property is fulfilled also. This case will be considered in detail later.

In the monotone case we can give the solution of Problem (P). We will use the following lemma (cf. [3]).

Let $Y=(Y_n)_{n=1}^{\infty}$ be a homogeneous Markov chain on $(\Omega,\mathcal{F},P)$ with a state space $(\mathbb{E},\mathcal{B})$ and let $p(\cdot\,;\cdot)$ denote the transition function, i.e. $p(e;B)=P(Y_{n+1}\in B\mid Y_n=e)$ for $B\in\mathcal{B}$. Let $f_0:\mathbb{E}\to\mathbb{R}$ be a bounded function. Define

$$\begin{aligned}&\Gamma=\{e\in\mathbb{E}: f_0(e)\geqslant\mathbb{P}_0f_0(e)\},\\&\sigma_\Gamma=\inf\{n:\ Y_n\in\Gamma\},\end{aligned}\tag{17}$$

where the operator $\mathbb{P}_0$ is defined by (6).

Lemma 3. *If the conditions*

(i) $p(e;\Gamma)=1$ *for* $e\in\Gamma$,

(ii) $\sigma_\Gamma<+\infty$ *a.s.*

are fulfilled, then σ_Γ *is the optimal stopping time for stopping of the Markov chain* Y *with the reward function* f_0.

Now, we show that in the monotone case the set Γ is just (i) "closed" and (ii) "realizable". The solution is given by the following theorem.

Theorem 2. *If the assumptions* (1)–(3) *hold and the sequence* $(d(k,x))_{k=-1}^{\infty}$ *changes sign once for each fixed x then a solution of Problem* (P) *exists and has the form*

$$\tau^*=\inf\{n:\ \xi_n=\max(\xi_1,\ldots,\xi_n)\text{ and }\xi_n\geqslant x_n\}\tag{18}$$

where the decision number x_n *is the least root of the equation* $c(n,x)=0$ *in* $[0,1]$, $n\in\mathbb{N}$. *The probability of winning using the optimal strategy is*

$$P(\text{win})=\sum_{n=1}^{\infty} p_n \sum_{k=1}^{n} P_n(k), \tag{19}$$

where

$$\begin{aligned} &P_n(1)=(1-x_1^n)/n, \\ &P_n(k+1)=\sum_{i=1}^{k}\frac{x_i^k}{k(n-k)}-\sum_{i=1}^{k}\frac{x_i^n}{n(n-k)}-\frac{x_{k+1}^n}{n}, \quad n-1\geq k\geq 1. \end{aligned} \tag{20}$$

Proof. First note that $\Gamma=\{e\in\mathbb{E}: f(e)\geq \mathbb{P}f(e)\}$ and that $(n,x)\in\Gamma$ is equivalent to $c(n,x)\geq 0$ (cf. (14) and (17)). Next suppose that $d(n,x)\geq 0$. Then $d(n,y)\geq 0$ for $y\geq x$ because $d(n,x)$ is a non-decreasing function on $[0,1]$ for n fixed. Taking into account the assumption that the sequence $d(k,y)$, for each y, changes sign at most once we obtain that $d(k,y)\geq 0$ for $k\geq n$ and $y\geq x$. Thus $c(k,x)\geq 0$ or equivalently $(k,x)\in\Gamma$ for $k\geq n$ and $y\geq x$ (cf. (14) once more).

Now we show that $c(n,x)\geq 0$ implies $c(n,y)\geq 0$ for $y\geq x$. Let $c(n,x)\geq 0$. Since $c(n,1)=\pi_n\geq 0$ takes there exists x having this property. Suppose there exists $y>x$ such that $c(n,y)<0$. Then $d(n,y)<0$ (because $(n,y)\in\Gamma$ otherwise) and thus $d(n,x)<0$ also.

Let $d(m,x)<0$ for $m=n,\dots,n+s$ and $d(n+s+1,x)\geq 0$ for a certain $s\geq 0$. From (14) we can write

$$c(n,x)=d(n,x)+xc(n+1,x). \tag{21}$$

Since $c(n,x)\geq 0$ and $d(n,x)<0$, then $c(n+1,x)>0$. If we repeat this simple conclusion iteratively for values $c(n+2,x),\dots,c(n+s,x)$ then we obtain

$$c(n+s,x)=d(n+s,x)+xd(n+s+1,x)+x^2d(n+s+2,x)+\cdots>0.$$

Recall that $d(k,y)$ is a non-decreasing function for each k. Moreover, $d(k,x)\geq 0$ for $k\geq n+s+1$. Thus $0<c(n+s,x)\leq c(n+s,y)$ and

$$c(n+s-1,x)\leq d(n+s-1,x)+xc(n+s,y)\leq c(n+s-1,y).$$

We repeat this operation several times obtaining $c(n,x)\leq c(n,y)$ contrary to the supposition. Thus we have $c(n,x)\geq 0$ for $x\geq x_n$ and $c(n,x)<0$ for $x<x_n$ where $x_n=\inf\{x: c(n,x)\geq 0\}$. Since $\mathbb{P}f(\delta)=0=f(\delta)$, the set Γ has the form

$$\Gamma=\{\delta\}\cup\bigcup_{n=1}^{\infty}\{n\}\times[x_n,1].$$

Now we show that the sequence $(x_n)_{n=1}^{\infty}$ is non-increasing. It suffices to show that $(n,x)\in\Gamma$ implies $(n+1,x)\in\Gamma$ for each $n\in\mathbb{N}$ and $x\in[0,1]$. Assume that $c(n,x)\geq 0$. If $d(n,x)\geq 0$ then $d(n+1,x)\geq 0$ also and hence $(n+1,x)\in\Gamma$. If $d(n,x)<0$ then from (21) we have $c(n+1,x)>0$ and thus $(n+1,x)\in\Gamma$.

Monotonicity of x_n combined with the fact that our chain "goes to the right and upward" imply that the assumption (i) of Lemma 3 holds. Since the chain Y attains the state δ a.s. then the assumption (ii) of this lemma holds too. Finally, we obtain that $\Gamma = \Delta$ and the stopping time τ^* given by (18) is optimal in Problem (P). Probability of the best choice using the optimal strategy can be calculated as follows:

$$P(\text{win}) = \sum_{n=1}^{\infty} P(\text{win} \mid N=n) p_n = \sum_{n=1}^{\infty} p_n \sum_{k=1}^{n} P(\text{win at } k\text{th obs.} \mid N=n).$$

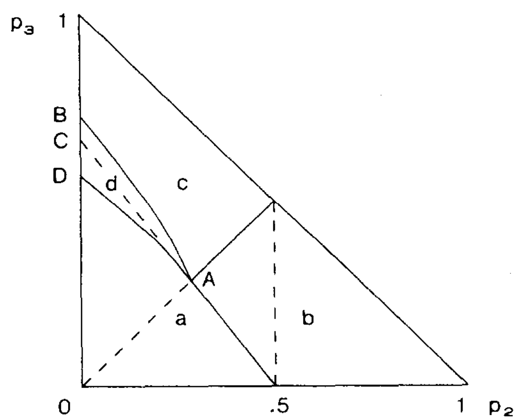
Gilbert and Mosteller [6, Theorem 4] proved that $P(\text{win at } k\text{th observation} \mid N=n) = P_n(k)$, where $P_n(k)$ is given by (20). Thus (19) is established and the proof is complete.

Now, for the purpose of illustrating the behaviour of $d(n,x)$ as well as the form of the stopping set we consider N distributed on $\{1,2,3\}$. Even such a simple example shows that Problem (P) is not completely resolved, because the assumption about $d(n,x)$ is not necessary.

Example 5. *Three-point distribution.* Let $p_n = 0$ for $n > 3$. Since $s(3,x) = f(3,x)$ (i.e. we must stop at each state $(3,x)$) then using the backward induction (by (12) and (8)) we can calculate $s(2,x)$ and $s(1,x)$. Taking into account (11) we easily obtain the general solution of this simple case. Let $x_2 = (p_3 - p_2)/(2p_3)$ (x_2 is a solution of $\mathbb{P}f(2,x) = f(2,x)$) and x_f, x_s be roots of equations

$$x_f\colon\quad 5p_3x^2 + (4p_2 - 2p_3)x + 2 - 4p_2 - 3p_3 = 0 \quad \text{in } [0,1],$$

$$x_s\colon\quad 3p_3x^2 + 2p_2x + 2p_3x_f^2 + 2(p_2 - p_3)x_f + 2 - 4p_2 - 3p_3 = 0 \quad \text{in } [0, x_f).$$



Equations of curves:
AB: $11p_3^2 + 14p_2p_3 + 3p_2^2 - 8p_3 = 0.$
AC: $4p_2 + 3p_3 - 2 = 0.$
AD: $7p_3^2 + 6p_2p_3 + p_2^2 - 4p_3 = 0.$

Fig. 1.

Depending on values of p_2 and p_3 ($p_1 = 1 - p_2 - p_3$) the following four optimal strategies are possible:

(a) Stop at the first observation.

(b) Stop at the first observation if its value exceeds x_f; otherwise stop at the next leader (if it will appear).

(c) Stop at ξ_1 if $\xi_1 \geqslant x_f$; if it was rejected then stop at the second observation if it is a leader greater than x_2. Here $x_2 < x_f$.

(d) Stop at the first observation if $\xi_1 \geqslant x_s$; otherwise stop at ξ_2 if it is a leader greater than x_2. Here $x_2 > x_s$.

The areas in which the above strategies are optimal are shown in Fig. 1 (naturally these areas are closed). Note that for $p_2 < p_3 < 2(1-2p_2)/3$ (the triangle OAC) the sequence $d(n, 0)$ changes sign twice and there are possible two different solutions: monotone (a) or non-monotone (d).

6. The special examples

Now we consider Problem (P) in detail when the period of observation has the geometric or the uniform distribution.

Example 6. *The geometric distribution with parameter p.* Let $p_k = pq^k$ for $k = 0, 1, 2, \ldots, 0 < p < 1$, $p + q = 1$. Then

$$\begin{aligned} d(k, x) &= pq^k - \int_x^1 \sum_{i=k+1}^{\infty} pq^i y^{i-(k+1)} \, \mathrm{d}y \\ &= pq^k \left(1 - \int_x^1 \sum_{i=k+1}^{\infty} (qy)^{i-(k+1)} q \, \mathrm{d}y \right) \\ &= pq^k (1 + \ln(p/(1-qx))), \end{aligned}$$

$$\begin{aligned} c(n, x) &= \left(1 + \ln \frac{p}{1-qx} \right) pq^n \sum_{k=n}^{\infty} (qx)^{k-n} \\ &= \left(1 + \ln \frac{p}{1-qx} \right) \frac{1}{1-qx} pq^n. \end{aligned}$$

Since, for each n, $c(n, x) \geqslant 0$ iff $1 + \ln(p/(1-qx)) \geqslant 0$, then $(n, x) \in \Gamma$ iff $x \geqslant (1-pe)/q$. Thus the stopping time

$$\tau^* = \inf\{n : \xi_n = \max(\xi_1, \ldots, \xi_n), \, \xi_n \geqslant (1-pe)/q\}$$

is optimal for this problem. We can write the definition of τ^* in a simpler form because the condition $\xi_n = \max(\xi_1, \ldots, \xi_n)$ is superfluous. Moreover, we can easily calculate the maximal probability when the optimal stopping rule is used. If we write $x_0 = \max((1-pe)/q, 0)$, i.e. $x_0 = 0$ if $p \geq e^{-1}$ and $x_0 = (1-pe)/q$ if $p < e^{-1}$, then using the convention $0^\circ = 1$ we can write

$$
\begin{aligned}
P(\text{win}) &= P(\tau^* \leq N,\ \xi_{\tau^*} = \max(\xi_1, \ldots, \xi_N)) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{m} P(N=m) P(\max(\xi_1, \ldots, \xi_{n-1}) < x_0, \quad \xi_n \geq x_0, \\
&\qquad \xi_n \geq \max(\xi_{n+1}, \ldots, \xi_m)) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{m} pq^m x_0^{n-1}(1 - x_0^{m-n+1})/(m-n+1) \\
&= \sum_{n=1}^{\infty} \sum_{m=n}^{\infty} pq^m x_0^{n-1}(1 - x_0^{m-n+1})/(m-n+1) \\
&= \sum_{n=1}^{\infty} pq^{n-1} x_0^{n-1} \left[\sum_{m=n}^{\infty} (q^{m-n+1} - (qx_0)^{m-n+1})/(m-n+1) \right] \\
&= \sum_{n=1}^{\infty} p(qx_0)^{n-1}(-\ln(1-q) + \ln(1-qx_0)) \\
&= \begin{cases} -p \ln p & \text{if } x_0 = 0, \\ e^{-1} & \text{if } x_0 > 0. \end{cases}
\end{aligned}
$$

The above results are summarized in the following theorem.

Theorem 3. *Assume that* (1) *and* (3) *hold and N has the geometric distribution with parameter p. Then,*

(a) *If $p \geq e^{-1}$ then $\tau^* = 1$ is the solution of Problem* (P) *and the probability that using this stopping rule we obtain the maximal ξ is equal to $-p \ln p$;*

(b) *If $p < e^{-1}$ then $\tau^* = \inf\{n: \xi_n \geq (1-pe)/q\}$ is the solution of Problem* (P) *and the above probability is equal to* e^{-1}.

In other words, the optimal stopping rule can be expressed as follows: stop the observation at the moment n at which the first "leader" occurs such that $\xi_n \geq (1-pe)/q$ if $p < e^{-1}$ and stop when the first ξ appears, otherwise.

It is rather interesting that in this example the more complex model has a solution which turns out to be simpler and more explicit. (Compare this with the solution of the full-information best choice problem with N nonrandom and known in [6] and with the solution of the no-information best choice problem when N has the geometric distribution in [7].) It is unexpected that in all "natural" situations (i.e. when $p < e^{-1}$) the probability of the correct stop is constant, equal to e^{-1}.

Example 7. *The uniform distribution on* $\{1, \ldots, n\}$. Let $p_k = 1/n$ for $k = 1, 2, \ldots, n$. Example 2 yields that this case is monotone and therefore the solution has the form given by Theorem 1. The optimal decision number x_{n-k} is a root of $c(n-k, x)=0$ in $[0, 1)$ (obviously $x_n=0$). This equation can be equivalently written as

$$\sum_{s=0}^{k} x^s = \sum_{s=0}^{k-1} x^s \sum_{j=1}^{k-s} (1-x^j)/j$$

or

$$1-x^{k+1} = -x^{k+1} \sum_{j=1}^{k} \frac{(1/x)^j-1}{j} - \sum_{j=1}^{k} \frac{x^j-1}{j}. \tag{22}$$

Notice that (22) does not depend on n. Denote $b_k = x_{n-k}$ (thus b_k is the optimal decision number for the $(n-k)$th observation). For $k=1$ we get $1-x=1+x$ and thus $b_1=0$. For $k=2$ after simplifying we get $5x^2+2x-1=0$ and $b_2=(\sqrt{6}-1)/5 \approx 0.2899$. The first column of Table 1 gives numerical results of b_k.

If we write $b_k = 1/(1+c(k)/(k-1))$ then $c(k)$ is approximately linear in $1/k$ and we obtain the asymptotic value $c \approx 2.1198$. The constant c is a solution of the equation

$$e^c - 1 = e^c \int_{-c}^{0} (e^y-1)/y \, dy - \int_0^c (e^y-1)/y \, dy$$

Table 1
The optimal decision numbers b_k when N is uniformly distributed*

k	b_k	appr.	k	b_k	appr.	k	b_k	appr.
1	0	0	21	0.9031	0.9036	41	0.9494	0.9495
2	0.2899	0.2934	22	0.9074	0.9078	42	0.9505	0.9507
3	0.4627	0.4700	23	0.9112	0.9117	43	0.9517	0.9518
4	0.5697	0.5760	24	0.9148	0.9152	44	0.9528	0.9529
5	0.6416	0.6467	25	0.9181	0.9185	45	0.9538	0.9539
6	0.6931	0.6972	26	0.9211	0.9215	46	0.9548	0.9549
7	0.7317	0.7350	27	0.9240	0.9243	47	0.9557	0.9558
8	0.7617	0.7645	28	0.9266	0.9269	48	0.9566	0.9567
9	0.7858	0.7880	29	0.9290	0.9293	49	0.9575	0.9576
10	0.8054	0.8073	30	0.9314	0.9316	50	0.9583	0.9584
11	0.8217	0.8233	31	0.9335	0.9338	51	0.9591	0.9592
12	0.8355	0.8369	32	0.9355	0.9358	52	0.9599	0.9600
13	0.8474	0.8486	33	0.9374	0.9377	53	0.9607	0.9607
14	0.8576	0.8587	34	0.9392	0.9394	54	0.9614	0.9615
15	0.8665	0.8675	35	0.9409	0.9411	55	0.9621	0.9621
16	0.8745	0.8753	36	0.9425	0.9327	56	0.9627	0.9628
17	0.8814	0.8822	37	0.9441	0.9442	57	0.9633	0.9634
18	0.8877	0.8884	38	0.9455	0.9456	58	0.9640	0.9641
19	0.8934	0.8940	39	0.9469	0.9470	59	0.9646	0.9647
20	0.8985	0.8991	40	0.9482	0.9483	60	0.9652	0.9652

Note: For $k \geqslant 20$ $b_{k-1} = 1-2.1198/k-0.24/k^2$ with the error less than 0.0001.

* The optimal strategy for N uniformly distributed on $\{1, \ldots, n\}$: accept ξ_k if it is a leader greater than b_{n-k}; otherwise reject ξ_k and await ξ_{k+1}.

which we can obtain from (22) by passing to the limit. This yields

$$b_k = 1 - \frac{c}{k+1} + o\left(\frac{1}{k}\right), \quad k \geq 2. \tag{23}$$

Thus the stopping rule

$$\tau_n^* = \min\left\{k\colon \xi_k = \max(\xi_1, \dots, \xi_k),\ \xi_k \geq 1 - \frac{c}{n-k+1}\right\}$$

(in which, instead of b_{n-k}, we take its approximation (23)) is asymptotically optimal. The second column of Table 1 compares the approximations of (23) with values precisely computed from (22).

The probability of winning calculated from (19) for $n = 1(1)10(5)20(10)60$ is shown in Table 2. $P(\text{win})$ is approximately linear in $1/n$ and we get an estimated asymptotic value 0.4352.

Table 2
The probability of winning when N is uniformly distributed on $\{1, \dots, n\}$

n	$P(\text{win})$	n	$P(\text{win})$	b	$P(\text{win})$
1	1	7	0.5193	30	0.4543
2	0.7500	8	0.5085	40	0.4495
3	0.6387	9	0.5001	50	0.4466
4	0.5855	10	0.4935	60	0.4447
5	0.5543	15	0.4738		
6	0.5338	20	0.4640	∞	0.4352

Note: For $n \geq 10$ $P(\text{win}) = 0.4352 + 0.568/n + 0.15/n^2$ with the error less than 0.0001.

7. Remarks

1. The no-information version of the best choice problem is known as the secretary (or beauty contest, dowry, marriage) problem. This problem with a random number of observations was posed first by Presman and Sonin [7]. They solved three special cases; where N is uniform, geometric or Poisson. For the uniform distribution from 1 to n $P(\text{win}) \to 2\,\mathrm{e}^{-2} \approx 0.2707$ as $n \to \infty$. A broad review of the secretary problem and its extensions has been made by Freeman [5].

2. The problems under full information and a random number of observations has been solved, as far as the author knows, only in some continuous time versions. Sakaguchi [8] and Bojdecki [1] independently considered the full-information best choice problem when observations appear according to a Poisson process and a decision about stopping must be made before a random moment T. They gave a solution for a fixed T. Bojdecki [2] also solved this problem when T is exponentially distributed. Example 6 corresponds to this case.

Acknowledgement

The author wishes to thank the referee for his helpful suggstions and comments.

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